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Sharp Estimates for Jacobi Matrices and Chain Sequences

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Chain sequences are positive sequences $\{a_n\}$ of the form $a_n = g_n(1 - g_{n-1})$ for a nonnegative sequence $\{g_n\}$. This concept has been introduced by Wall in connection with continued fractions. These sequences are very useful in determining the support of orthogonality measure for orthogonal polynomials. Equivalently, they can be used for localizing spectra of Jacobi matrices associated with orthogonal polynomials through the recurrence relation.

We derive sharp estimates for chain sequences which in turn give sharp estimates for the norms of Jacobi matrices. We also give applications to unbounded essentially self-adjoint Jacobi matrices. In particular, we show how to determine whether their spectrum admits gaps around 0, and derive some integrability properties of the spectral measure. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

One of the main problems of the theory of orthogonal polynomials is the following. Given a three-term recurrence formula which the polynomials satisfy find a measure with respect to which the polynomials are orthogonal. The existence is guaranteed by the Favard theorem; however, the nature of this measure is not described by the theorem. The recurrence relation determines the Jacobi matrix J. A problem now can be reformulated into: assuming J is essentially self-adjoint find its spectral resolution of the identity.

The first problem is to locate the spectrum of J. Of course, there are many instances of Jacobi matrices for which this is known explicitly, along with spectral measure (see [5]). Then one may expect to be able to identify spectra for small perturbations of the known matrices.

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One of the most useful tools for locating the interval of orthogonality for orthogonal polynomials, or the interval containing the spectrum of the corresponding Jacobi matrix, is given by the so-called chain sequences. These are sequences $\{a_n\}_{n=1}^{\infty}$, for which there exist a sequence $\{g_n\}_{n=0}^{\infty}$ such that $0 \leqslant g_n \leqslant 1$ and

$$a_n = g_n(1 - g_{n-1}), \qquad n \geqslant 1.$$

These sequences were introduced by Wall in his Monograph on continued fractions. He proved the following (see [10] or [3, Example 5.13, p. 100]).

Theorem A (Wall). *Given a sequence* $\{a_n\}_{n=1}^{\infty}$. *Let* $J = \{J(m,n)\}_{n,m=0}^{\infty}$ *be the Jacobi matrix with entries defined as*

$$J_{n,m} = \begin{cases} \sqrt{a_{n+1}} & \text{for } m = n+1, \\ \sqrt{a_n} & \text{for } m = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then J *is a contraction on the Hilbert space* $\ell^2(\mathbb{N})$ *if and only if* $\{a_n\}_{n=1}^{\infty}$ *is a chain sequence.*

In applications we will also make use of the following (see [2] or [3], Theorem IV.2.1, p. 108).

Theorem B (Chihara). *Assume a sequence of orthogonal polynomials which satisfies*

$$x p_n(x) = \lambda_{n+1} p_{n+1}(x) + \beta_n p_n(x) + \lambda_n p_{n-1}(x), \qquad n \geqslant 0,$$

where $\lambda_0 = 0$, $\lambda_n > 0$ *and* $\beta_n \in \mathbb{R}$. *Let a number* $a \in \mathbb{R}$ *satisfy*

(i) $a < \beta_n$, *for* $n \geqslant 0$.
(ii) *The numbers* $\frac{\lambda_n^2}{(\beta_{n-1}-a)(\beta_n-a)}$, *for* $n \geqslant 1$, *form a chain sequence.*

Then there is an orthogonality measure μ *such that* $\operatorname{supp} \mu \subset [a, +\infty]$. *The converse is also true.*

We will study chain sequences in order to get sharp estimates for bounded Jacobi matrices in view of Theorem A. Also, what is surprising, we will apply them to study the spectra of unbounded Jacobi matrices as well, using this time Theorem B. The main results are contained in Section 2, where sharp estimates are obtained for chain sequences.

2. SHARP ESTIMATES FOR CHAIN SEQUENCES

The greatest constant chain sequence is $a_n = \frac{1}{4} = \frac{1}{2}(1 - \frac{1}{2})$. The problem arises by how much a chain sequence can exceed $\frac{1}{4}$. The answer is given by the next two theorems. The first theorem follows from [4, Theorem 2.2] but our proof is different and will be used to show a sharper result stated in Theorem 2.

THEOREM 1. *Let $\{a_n\}_{n=0}^{\infty}$ be a chain sequence such that*

$$a_n \geqslant \frac{1}{4} + \frac{c}{16n^2}$$

for $n \geqslant N$. Then $c \leqslant 1$.

Proof. Let $\varepsilon > 0$. By replacing the constant c with $c' = c - \varepsilon$ and replacing the number N with an appropriately bigger number N' we may assume that

$$a_n \geqslant \frac{1}{4} + \frac{c'}{4(4n^2 - 1)}$$

for $n \geqslant N'$. Let $d = \frac{c'}{4}$. Then we have

$$a_n \geqslant \frac{1}{4} + \frac{d}{4n^2 - 1}.$$

It suffices to show that $d \leqslant \frac{1}{4}$. By assumptions there exists a sequence g_n, such that $0 \leqslant g_n \leqslant 1$ and

$$g_n(1 - g_{n-1}) \geqslant \frac{1}{4} + \frac{d}{4n^2 - 1}. \tag{1}$$

Since $g_n(1 - g_{n-1}) \geqslant \frac{1}{4}$ and $g_{n-1}(1 - g_{n-1}) \leqslant \frac{1}{4}$ we get that the sequence g_n is nondecreasing and therefore its limit is $\frac{1}{2}$. Hence, we can write g_n in the form $g_n = \frac{1-\delta_n}{2}$ for $0 \leqslant \delta_n \leqslant 1$. Substituting this into (1) gives

$$\delta_{n-1} - \delta_n - \delta_{n-1}\delta_n \geqslant \frac{d}{n^2 - \frac{1}{4}}. \tag{2}$$

The latter implies

$$\delta_{n-1} - \delta_n \geqslant \frac{d}{n - \frac{1}{2}} - \frac{d}{n + \frac{1}{2}}.$$

Adding up the terms results in

$$\delta_n \geqslant \frac{d}{n+\frac{1}{2}} \qquad \text{for } n \geqslant N'. \tag{3}$$

Let D be the greatest positive number such that

$$\delta_n = \frac{D}{n+\frac{1}{2}} + \varepsilon_n, \tag{4}$$

where $\varepsilon_n \geqslant 0$ and $n \geqslant N'$. By (3) we have $D \geqslant d$. Substituting (4) into (2) yields

$$\varepsilon_{n-1} - \varepsilon_n - \varepsilon_{n-1}\varepsilon_n \geqslant \frac{D^2 - D + d}{n^2 - \frac{1}{4}} + \frac{D\varepsilon_n}{n-\frac{1}{2}} + \frac{D\varepsilon_{n-1}}{n+\frac{1}{2}}. \tag{5}$$

In particular, we have

$$\varepsilon_{n-1} - \varepsilon_n \geqslant \frac{D^2 - D + d}{n^2 - \frac{1}{4}} = (D^2 - D + d)\left(\frac{1}{n-\frac{1}{2}} - \frac{1}{n+\frac{1}{2}}\right). \tag{6}$$

Adding up the terms gives

$$\varepsilon_n \geqslant \frac{D^2 - D + d}{n+\frac{1}{2}}.$$

By the definition of D we must have

$$0 \geqslant D^2 - D + d = \left(D - \frac{1}{2}\right)^2 + d - \frac{1}{4}.$$

Hence $d \leqslant \frac{1}{4}$. ∎

Remark 1. The estimate $c \leqslant 1$ is sharp because the numbers

$$a_n = \frac{1}{4} + \frac{1}{4(4n^2 - 1)} = \frac{n}{2n+1}\left(1 - \frac{n-1}{2n-1}\right)$$

form a chain sequence.

Remark 2. From the proof we get the estimate

$$\frac{1}{2} - \sqrt{\frac{1}{4} - d} \leqslant D \leqslant \frac{1}{2} + \sqrt{\frac{1}{4} - d}.$$

When $d = \dfrac{1}{4}$ we obtain $D = \dfrac{1}{2}$.

Proposition 1. *The sequence*

$$a_n = \frac{1}{4} + \frac{1}{4(4n^2-1)}, \qquad n \geqslant 1$$

is a maximal chain sequence, i.e. for any chain sequence $\{b_n\}_{n=1}^{\infty}$ *if* $b_n \geqslant a_n$ *for every* $n \geqslant 1$ *then* $b_n = a_n$ *for every* $n \geqslant 1$.

Proof. Let b_n be a chain sequence such that $b_n \geqslant a_n$ for $n \geqslant 1$. Hence, there are numbers $0 \leqslant g_n \leqslant 1$ for $n \geqslant 0$ such that

$$g_n(1-g_{n-1}) \geqslant \frac{1}{4} + \frac{1}{4(4n^2-1)}$$

for every $n \geqslant 1$. From the proof of Theorem 1 we know that g_n is of the form $g_n = \frac{1-\delta_n}{2}$, where $\delta_n \geqslant 0$. By Remark 2 and by (4) we get

$$\delta_n = \frac{1}{2n+1} + \varepsilon_n$$

for some nonnegative sequence ε_n. We have

$$g_0 = \frac{1-\delta_0}{2} = -\frac{\varepsilon_0}{2}.$$

Thus $\varepsilon_0 = 0$. Moreover, by (6) we get

$$\varepsilon_{n-1} - \varepsilon_n \geqslant 0.$$

Hence $\varepsilon_n = 0$ for any $n \geqslant 0$. Consequently, $\delta_n = \dfrac{1}{2n+1}$ and

$$b_n = g_n(1-g_{n-1}) = \frac{1}{4} + \frac{1}{4(4n^2-1)} = a_n. \quad \blacksquare$$

We will study now by how much a chain sequence may exceed $\frac{1}{4} + \frac{1}{16n^2}$ for $n \geqslant N$. This answers a question posed in [4, Remark, p. 626].

Theorem 2. *Let* a_n *be a chain sequence satisfying*

$$a_n \geqslant \frac{1}{4} + \frac{1}{16n^2} + \frac{c}{n^2 \log^{\alpha} n}$$

for $n \geqslant N$ *with* $c > 0$. *Then* $\alpha \geqslant 2$.

Proof. By assumptions there exists a sequence g_n, such that $0 \leqslant g_n \leqslant 1$ and $a_n = g_n(1-g_{n-1})$. By replacing the number N with an appropriately bigger number N' we may assume that

$$g_n(1 - g_{n-1}) \geqslant \frac{1}{4} + \frac{1}{4(4n^2 - 1)} + \frac{\varepsilon_n}{4(4n^2 - 1)} \tag{7}$$

for $n \geqslant N'$, where

$$\varepsilon_n = \frac{c}{\log^\alpha n}. \tag{8}$$

By the proof of Theorem 1 we can write $g_n = \frac{1-\delta_n}{2}$, where $\delta_n \geqslant 0$, for $n \geqslant N'$. Moreover, using Remark 2 and (4) gives that

$$\delta_n \geqslant \frac{1}{2n+1}$$

for $n \geqslant N'$. Therefore, we may write

$$\delta_n = \frac{1 + u_n}{2n+1}$$

for some sequence u_n with nonnegative terms for $n \geqslant N'$.

Now we substitute this into (7) to obtain after obvious simplifications the following:

$$u_{n-1} - u_n \geqslant \frac{\varepsilon_n}{2n} + \frac{u_{n-1}u_n}{2n}, \qquad n \geqslant N'. \tag{9}$$

By summing up we get

$$u_n \geqslant \sum_{k=n+1} \frac{\varepsilon_k}{2k} \tag{10}$$

for $n \geqslant N'$. On the other hand (9) implies

$$u_{n-1} - u_n \geqslant \frac{u_{n-1}u_n}{2n}, \qquad n \geqslant N'.$$

Thus,

$$\frac{1}{u_n} - \frac{1}{u_{n-1}} \geqslant \frac{1}{2n}, \qquad n \geqslant N'.$$

Hence,

$$\frac{1}{u_n} \geqslant \frac{1}{2}\log n + C$$

for $n \geqslant N'$ and some real constant C. This and (10) imply

$$\sup_n\ (\log n) \sum_{k=n}^{\infty} \frac{\varepsilon_k}{k} < +\infty. \tag{11}$$

Now (8) yields $\alpha \geqslant 2$. ∎

Remark 3. The estimate $\alpha \geqslant 2$ is sharp. Indeed, it is not hard to show that taking $g_0 = 0,\ g_1 = \frac{1}{2}$ and

$$g_n = \frac{n}{2n+1} - \frac{1}{2(2n+1)\log n}, \qquad n \geqslant 2$$

gives rise to the chain sequence $a_n = g_n(1 - g_{n-1})$ which satisfies the assumptions of Theorem 2 with $\alpha = 2$.

Remark 4. Theorem 2 gives sharper estimates than the one given in [4, Theorem 2.2], which states that if

$$a_n = \frac{1}{4} + \frac{1 + e_n}{16n^2} \tag{12}$$

with $e_n \geqslant 0$ for large n and $\sum (e_n/n) = +\infty$, then a_n cannot be a chain sequence. Indeed, let $e_n = \log^{-\alpha}(n+2)$, where $1 < \alpha < 2$. Then $\sum\ (e_n/n) < +\infty$, but by Theorem 2 the sequence defined by (12) is not a chain sequence.

Combining Theorem A and Theorem 1 implies the following.

THEOREM 3. *Let J be a Jacobi matrix with entries*

$$J(n,m) = \begin{cases} \lambda_n & \textit{for } m = n+1, \\ \lambda_{n+1} & \textit{for } m = n-1, \\ 0 & \textit{otherwise.} \end{cases} \tag{13}$$

Assume

$$\lambda_n \geqslant \frac{1}{2} + \frac{C}{16n^2}$$

for n sufficiently large. Assume J is a contraction on the Hilbert space $\ell^2(\mathbb{N})$. Then $C \leqslant 1$.

Remark. One may try to prove Theorem 13 directly. Indeed, assuming that $C > 1$ one has to show that

$$\|J\| = \sup\{(Jx, x) \mid (x,x) \leqslant 1\} > 1.$$

We may assume that $\lambda_n \to 1/2$. Then J is a compact perturbation of the constant Jacobi matrix J_0, where

$$J(n,m) = \begin{cases} \frac{1}{2} & \text{for } m = n+1, \\ \frac{1}{2} & \text{for } m = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that $\sigma(J_0) = [-1, 1]$. Thus by the Weyl theorem the continuous spectrum of J coincides with $[-1, 1]$. Hence, the number $\|J\| > 1$ is an eigenvalue of J. Therefore, if one wants to show that

$$\sup\{(Jx, x) \mid (x, x) \leqslant 1\} > 1,$$

he must find an eigenvector of J, or a vector pretty close to it. As we know it is not an easy task, even if we can guess the asymptotic behaviour of the coordinates of this eigenvector.

By Proposition 1 we obtain the following.

Corollary 1. *Let J be a Jacobi matrix with entries corresponding to the sequence*

$$\lambda_n = \sqrt{\frac{1}{4} + \frac{1}{4(4n^2 - 1)}}$$

according to (13). *Then J is the maximal Jacobi matrix which is a contraction on $\ell^2(\mathbb{N})$, i.e. any matrix J' corresponding to $\lambda_n' \geqslant \lambda_n$ for all $n \geqslant 1$ is a contraction only if $\lambda_n' = \lambda_n$ for all $n \geqslant 1$.*

3. APPLICATIONS TO UNBOUNDED JACOBI MATRICES

Theorem 1 can be applied also in the context of unbounded Jacobi matrices. Assume that J is defined by (13), where the coefficients λ_n may be positive and unbounded. We may also assume that the corresponding matrix J is essentially self-adjoint on the space of finitely supported sequences in $\ell^2(\mathbb{N})$. By Hamburger's theorem this property is equivalent to the fact that the eigenvalue equation

$$\lambda_{n+1}x_{n+1} + \lambda_n x_{n-1} = 0, \qquad n \geqslant 1,$$

admits solutions $x = \{x_n\}_{n=0}^{\infty}$ which are not square summable (see [1, Problem 10, p. 84] or [7, Proposition 5.13]). One can compute easily that such solutions exist if and only if

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_1\lambda_3\ldots\lambda_{2n-1}}{\lambda_2\lambda_4\ldots\lambda_{2n}}\right)^2=\infty \qquad \text{or} \qquad \sum_{n=1}^{\infty}\left(\frac{\lambda_2\lambda_4\ldots\lambda_{2n}}{\lambda_3\lambda_5\ldots\lambda_{2n+1}}\right)^2=\infty. \tag{14}$$

We want to decide if the interval $(-a, a)$ is disjoint from the spectrum of J and we would like to determine the largest a possible.

Consider the sequence of polynomials $p_n(x)$ defined recursively by $p_{-1}=0$, $p_0=1$, and

$$x\,p_n(x)=\lambda_{n+1}\,p_{n+1}+\lambda_n\,p_{n-1}, \qquad n\geqslant 0. \tag{15}$$

By Favard's theorem there is a probability measure $d\mu(x)$ such that

$$\int_{-\infty}^{\infty} p_n(x)\,p_m(x)\,d\mu(x)=\delta_n^m.$$

This measure is unique, since we have assumed that J is essentially self-adjoint. Moreover, the operator corresponding to J is unitarily equivalent to multiplication by the variable x on the Hilbert space $L^2(\mathbb{R}, d\mu)$. Therefore, the spectrum of J coincides with $\operatorname{supp}(\mu)$. Thus, our aim will be to find

$$a=\inf\{x\geqslant 0 \mid x\in \operatorname{supp}\mu\}.$$

By (15) the polynomials p_{2n} are even functions while p_{2n+1} are odd ones. Hence, the orthogonality measure μ is symmetric about the origin. Moreover, observe that $q_n(y)=p_{2n}(\sqrt{y})$ is a polynomial of nth degree satisfying $q_0=0$ and

$$yq_n=\lambda_{2n+1}\lambda_{2n+2}q_{n+1}+[\lambda_{2n}^2+\lambda_{2n+1}^2]q_n+\lambda_{2n-1}\lambda_{2n}q_{n-1} \tag{16}$$

for $n\geqslant 0$ with the convention $\lambda_0=0$. The polynomials q_n are orthogonal on the positive half-axis with respect to the measure $dv(y)=2d\mu(\sqrt{y})$ for $y>0$ and $v(\{0\})=\mu(\{0\})$. Now we see that

$$a=\inf\{\sqrt{y} \mid y\in \operatorname{supp} v\}.$$

Thus, the number a^2 is located to the left of $\operatorname{supp} v$. By Theorem B this is equivalent to the fact that the sequence

$$a_n=\frac{\lambda_{2n-1}^2\lambda_{2n}^2}{(\lambda_{2n-2}^2+\lambda_{2n-1}^2-a^2)(\lambda_{2n}^2+\lambda_{2n+1}^2-a^2)} \tag{17}$$

is a chain sequence. Moreover, a_n is a maximal chain sequence if and only if the integral $\int_0^\infty \frac{1}{y-a^2}\,dv(y)$ is infinite (see [8, Theorem 1, or 9, Theorem 5]).

This in turn is equivalent to the fact that $\int_{-\infty}^{\infty} \frac{1}{x^2-a^2}\, d\mu(x) = \infty$. Observe that $\lambda_n \to \infty$ and $\lambda_n/\lambda_{n-1} \to 1$ when n tends to infinity. Then the sequence in (17) tends to 1/4. That is why the results of Section 1 can be useful here.

Let us turn now to examples.

EXAMPLE 1. Assume that the numbers λ_n satisfy $\lambda_{2n} = \lambda_{2n+1}$. Then J is essentially self-adjoint. By (17) we have for $a = 0$,

$$a_n = \begin{cases} \frac{1}{2} & \text{for } n = 1, \\ \frac{1}{4} & \text{for } n \geqslant 2. \end{cases}$$

It can be checked easily that the sequence a_n is a maximal chain sequence. Hence, we cannot get a chain sequence in (17) by taking positive a. Therefore, 0 belongs to the spectrum of J. Moreover, by maximality of a_n we get

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\, d\mu(x) = +\infty.$$

If the first series in (14) is divergent, the number 0 is not an eigenvalue. Hence 0 is an accumulation point of the spectrum of J.

EXAMPLE 2. Let J be associated with the sequence $\lambda_n = n$. According to (17) we have to study the sequence

$$a_n = \frac{(2n-1)^2(2n)^2}{[(2n-2)^2 + (2n-1)^2 - a^2][(2n)^2 + (2n+1)^2 - a^2]}.$$

It is not hard to compute that

$$a_n = \frac{1}{4} + \frac{a^2+1}{16n^2} + O(n^{-3}).$$

Thus by Theorem 1, the numbers a_n form a chain sequence only for $a = 0$. Furthermore, the spectrum of the Jacobi matrix J is not isolated from the point 0. This particular operator is pretty well known. Its spectrum coincides with the whole real line and the corresponding measure μ is absolutely continuous. Moreover, we have

$$a_n \leqslant b_n = \frac{n^2}{4n^2-1}$$

and the sequences a_n and b_n do not coincide. Thus, a_n is not a maximal chain sequence. Hence, if μ denotes the orthogonality measure then

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\, d\mu(x) < +\infty.$$

It is worthwhile observing that the polynomials associated with the matrix J are the limit cases, when $k \to 1$ of the polynomials studied by Stieltjes and Carlitz (see [3, (9.3), (9.4), p. 193]).

The next example has been studied by Moszyski [6].

EXAMPLE 3. Let J be associated with the sequence $\lambda_{2n-1} = \lambda_{2n} = n$. According to (17) we have to study the sequence

$$a_n = \frac{n^4}{[(n-1)^2 + n^2 - a^2][n^2 + (n+1)^2 - a^2]}.$$

One can compute easily that

$$a_n = \frac{1}{4} + \frac{a^2}{4n^2} + O(n^{-3}).$$

Therefore, a_n can be a chain sequence only for $a \leqslant \frac{1}{2}$. Let $a = \frac{1}{2}$. Then

$$a_n = \frac{16n^4}{(8n^2+3)^2 - 64n^2} \leqslant \frac{1}{4} + \frac{1}{4(4n^2-1)}.$$

Since the right-hand side is a chain sequence (see Proposition 1) a_n also is a chain sequence. Summarizing we obtain that the interval $(-1/2, 1/2)$ is disjoint from the spectrum of J and this is the largest interval with that property, which means that $\pm\frac{1}{2} \in \sigma(J)$. This fact has been proved in [6] by estimating below the quadratic form of the operator J. Moreover a_n is not a maximal chain sequence which implies that the orthogonality measure μ corresponding to J satisfies

$$\int_{-\infty}^{\infty} \frac{1}{x^2 - \frac{1}{4}}\, d\mu(x) < +\infty.$$

This implies in particular that the numbers $\pm\frac{1}{2}$ are not eigenvalues of J.

EXAMPLE 4. Let J be associated with the sequence $\lambda_{2n-1} = \lambda_{2n} = \sqrt{n}$. In view of (17) we obtain the sequence

$$a_n = \frac{n^2}{(2n-1-a)(2n+1-a)} \geqslant \frac{n^2}{4n^2-1} = \frac{1}{4} + \frac{1}{4(4n^2-1)}.$$

Hence, by Proposition 1 the numbers a_n form a chain sequence only for $a = 0$. Moreover, by [8, Theorem 1] the corresponding orthogonality measure has no finite moment of order -2. Moreover, it can be checked directly that 0 is not an eigenvalue, hence it is an accumulation point of the spectrum.

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REFERENCES

1. N. I. Akhiezer, "The Classical Moment Problem," Hafner Publ. Co., New York, 1965.
2. T. Chihara, Chain sequences and orthogonal polynomials, *Trans. Amer. Math. Soc.* **104** (1962), 1–16.
3. T. Chihara, "An Introduction to Orthogonal Polynomials," Mathematics and its Applications, Vol. 13, Gordon and Breach, New York/London/Paris, 1978.
4. T. Chihara, Spectral properties of orthogonal polynomials, *Trans. Amer. Math. Soc.* **170** (1982), 623–639.
5. R. Koekoek, R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report 98-17, TU Delft, 1998.
6. M. Moszyǹski, Spectral properties of some Jacobi matrices with double weights, preprint.
7. B. Simon, The classical moment problem as a self-adjoint finite difference operator, *Adv. Math.* **137** (1998), 82–203.
8. R. Szwarc, Chain sequences and compact perturbations of orthogonal polynomials, *Math. Z.* **217** (1994), 57–71.
9. R. Szwarc, Chain sequences, orthogonal polynomials and spectral radii of Jacobi matrices, *J. Approx. Theory* **92** (1998), 59–73.
10. H. S. Wall, "Analytic Theory of Continued Fractions," D. van Nostrand Co., New York, 1948.